Fall 2016 GEOL0350–GeoMath Final Exam

1 Wavy Gravy

The seismic wave equation is:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \left(\frac{\lambda}{\rho} + 2\beta^2\right) \nabla(\nabla \cdot \mathbf{u}) - \beta^2 \nabla \times \nabla \times \mathbf{u}. \tag{1}$$

Where $\mathbf{u}$ is the displacement vector field, and λ and β are measures of elasticity (Lamé parameters). If we take the curl and divergence of this equation, we find, respectively,

$$\frac{\partial^2 \nabla \times \mathbf{u}}{\partial t^2} = \beta^2 \nabla^2 \left(\nabla \times \mathbf{u}\right), \qquad \frac{\partial^2 \nabla \cdot \mathbf{u}}{\partial t^2} = \left(\frac{\lambda}{\rho} + 2\beta^2\right) \nabla^2 \left(\nabla \cdot \mathbf{u}\right). \tag{2}$$

These two equations are wave equations for the curl of the displacement (S-waves) and the divergence of the displacement (P-waves). You can assume λ, ρ, β are positive.

1.1 Diverge and Separate

Use the method of separation of variables to separate out a time dependent part of the displacement divergence ($T_\cdot(t)$) from a space dependent part ($X_\cdot(x, y, z)$). What equation does each $X(x, y, z)$ solve? (You don't have to split up the spatial coordinates.)

Plugging into the divergence equation, and dividing by TX,

$$\frac{\partial^2 T_\cdot(t) X_\cdot(x,y,z)}{\partial t^2} = \left(\frac{\lambda}{\rho} + 2\beta^2\right) \nabla^2 \left(T_\cdot(t) X_\cdot(x,y,z)\right), \tag{3}$$

$$\frac{1}{T_\cdot(t)} \frac{\partial^2 T_\cdot(t)}{\partial t^2} = -\omega^2 = \frac{1}{X(x,y,z)} \left(\frac{\lambda}{\rho} + 2\beta^2\right) \nabla^2 \left(X_\cdot(x,y,z)\right), \tag{4}$$

$$0 = \frac{\omega^2}{\left(\frac{\lambda}{\rho} + 2\beta^2\right)} X_\cdot(x,y,z) + \nabla^2 \left(X_\cdot(x,y,z)\right) \tag{5}$$

Thus, X solves the Helmholtz equation, and the general solution is a superposition over many possible frequencies.

1.2 Curl and Separate

Use the method of separation of variables to separate out a time dependent part of these equations ($T_\times(t)$) from a space dependent part ($X_\times(x, y, z)$).

Plugging into the curl equation, and dividing by TX,

$$\frac{\partial^2 T_\times(t) X_\times(x,y,z)}{\partial t^2} = \beta^2 \nabla^2 \left(T_\times(t) X_\times(x,y,z)\right), \tag{6}$$

$$0 = \frac{\omega^2}{\beta^2} X_\times(x,y,z) + \nabla^2 \left(X_\times(x,y,z)\right) \tag{7}$$

Thus, X solves the Helmholtz equation, and the general solution is a superposition over many possible frequencies.

1.3 Guess and Check: Wavenumber and Frequency

Assume that both the divergence and the curl have $T(t) \propto e^{i\omega t}$ with frequency ω, and $X(x, y, z) \propto e^{\pm ikx}$ with wavenumber $\pm k$ (here we are considering only one spatial direction for simplicity). Use (2) to write two equations relating k to ω (these are called the dispersion relations).

$$\omega^2 = \left(\frac{\lambda}{\rho} + 2\beta^2\right) k^2, \tag{8}$$

$$\omega^2 = \beta^2 k^2. \tag{9}$$

1.4 Phase Speed

The *phase speed* of the waves is $c = \omega/k$. Find the phase speed for the divergence and curl of the displacement. Which has the faster phase speed?

$$\frac{\omega}{k} = \sqrt{\frac{\lambda}{\rho} + 2\beta^2}, \tag{10}$$

$$\frac{\omega}{k} = \beta. \tag{11}$$

As the parameters are all positive, the divergent wave is faster since the first relation is always greater than β.

1.4.1 Initiate One or the Other

A seismic disturbance $\mathbf{u}_0$ will generally trigger both divergence $\nabla \cdot \mathbf{u}_0$ and curl $\nabla \times \mathbf{u}_0$ initial conditions that will propagate away as waves. What kind of mathematical form for $\mathbf{u}_0$ would possess only divergence waves? What kind of mathematical form for $\mathbf{u}_0$ would have only curl waves?

If the initial disturbance is a pure gradient $\mathbf{u}_0 = \nabla\phi$, then it will excite only a divergence wave (as curl of gradient is zero). Likewise, if the initial disturbance is a pure curl, $\mathbf{u}_0 = \nabla \times \psi$, then the wave will be a pure curl wave.

2 PDF PDQ

Three different probability distribution functions are being considered to explain the results of an experiment, which has independent measurements of x repeatedly. They are:

$$\rho_u(x;\sigma,\mu) = \frac{\mathcal{H}(\sqrt{3}\sigma - |x-\mu|)}{2\sqrt{3}\sigma}, \quad \rho_e(x;\sigma,\mu) = \frac{1}{\sigma\sqrt{2}}e^{-\sqrt{2}|x-\mu|/\sigma}, \quad \rho_n(x;\sigma,\mu) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)} \tag{12}$$

Where $\mathcal{H}$ is the Heaviside function (zero when it's argument is negative and 1 when positive). The three distributions have the same mean μ and standard deviation σ. Half of the experimental data fall above the mean and half below. Here we consider only the half that are above the mean.

Hints:

$$\int_{\mu}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0.5, \quad \int_{\mu}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x = 0.5, \quad \int_{\mu}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x = 0.5 \tag{13}$$

$$\int_{\mu+\sigma}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0.21, \quad \int_{\mu+\sigma}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x \approx 0.12, \quad \int_{\mu+\sigma}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x \approx 0.16, \tag{14}$$

$$\int_{\mu+2\sigma}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0, \quad \int_{\mu+2\sigma}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x \approx 0.03, \quad \int_{\mu+2\sigma}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x \approx 0.02, \tag{15}$$

$$\int_{\mu+3\sigma}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0, \quad \int_{\mu+3\sigma}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x \approx 0.007, \quad \int_{\mu+3\sigma}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x \approx 0.001. \tag{16}$$

2.1 Flatlining

Suppose one datum falls between μ and $\mu+\sigma$ and one datum falls between $\mu+\sigma$ and $\mu+2\sigma$. Which distribution is closest to the histogram?

The uniform distribution has 21% likelihood of being between $\mu+\sigma$ and $\mu+2\sigma$ and 29% between μ and $\mu+\sigma$, so it is closest. The others predict fewer between $\mu+\sigma$ and $\mu+2\sigma$.

2.2 Gambler's Fallacy

If the true distribution were normal (ρ_n), there are roughly twice as many data expected to fall between μ and $\mu+\sigma$ as between $\mu+\sigma$ and ∞. Suppose the first two experiments out of three fall between μ and $\mu+\sigma$. Which is the most likely next range: between μ and $\mu+\sigma$ or $\mu+\sigma$ and ∞?

Even though it seems like the next one needs to go into the $\mu+\sigma$ and ∞ range to fit the distribution, every experiment is independent, so μ to $\mu+\sigma$ is still the most likely range.

2.3 Combine the Probabilities

After two experiments, we decide to evaluate the following likelihood, based on the two results ($x > \mu+2\sigma, x > \mu$). If p(A) is the probability of experiment 1 being $x > \mu+2\sigma$, and p(B) is the probability of experiment 2 being $x > \mu$, what are the probabilities of any of the two $p(A+B)$ being true and both of the two $p(AB)$, *for each of the three distributions*? Can you eliminate any of the distributions by these results?

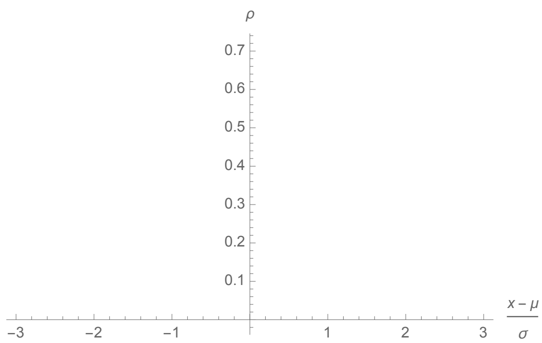
$$\begin{aligned} p(AB) &= p_A(B) \cdot p(B) = p(A)p(B), & p(A+B) &= p(A) + p(B) - p(AB), && (17)\\ p_u(AB) &= 0 \cdot 0.5 = 0, & p_u(A+B) &= 0 + 0.5 = 0.5, && (18)\\ p_e(AB) &= 0.03 \cdot 0.5 = 0.015, & p_e(A+B) &= 0.03 + 0.5 - 0.015 = 0.515, && (19)\\ p_n(AB) &= 0.02 \cdot 0.5 = 0.01, & p_n(A+B) &= 0.02 + 0.5 - 0.01 = 0.51, && (20)\\ & && && (21) \end{aligned}$$

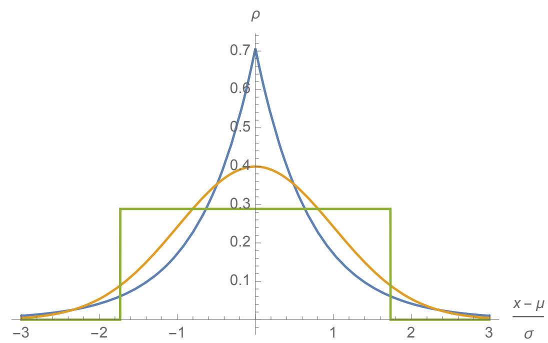
The uniform distribution is eliminated by these results.

2.4 Graph em linear!

Plot the probability distributions given between $-3\sigma + \mu$ and $+3\sigma + \mu$ on the following (linear) axes.

We draw the pdfs, noting their relative values and shapes.

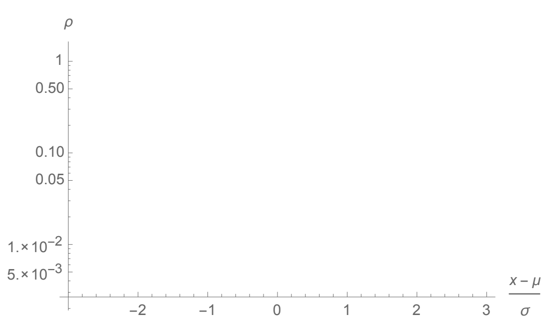


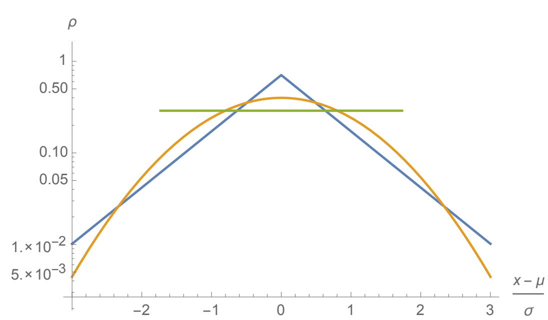


2.5 Graph em log!

Plot the probability distributions given between $-3\sigma+\mu$ and $+3\sigma+\mu$ on the following (semi-log) axes.

We draw the pdfs, noting their relative values and shapes. (Note, you do not need to plot ρ_u where it is zero).





3 Erosion by a Pebble in a Basin

Over time, a basin has formed with the shape (polar coordinates r, θ, dimensionless units):

$$h(r) = r^4 - 2r^2 + 1. \tag{22}$$

This basin has formed like this because there is a pebble in the basin, which is driven by the wind to have the following dynamics,

$$\dot{r} = -\nabla h(r) = -4r^3 + 4r^2, \tag{23}$$

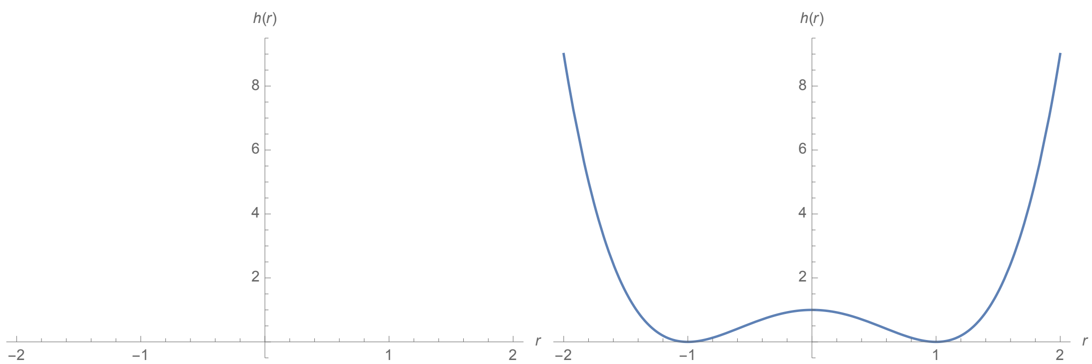
$$\dot{\theta} = 1. \tag{24}$$

(Note that negative values of r are used, so that a complete cross-section can be drawn, but of course these are just equivalent to shifting θ by π.)

3.1 Fill in the hole!

Plot the shape of the basin $h(r)$.

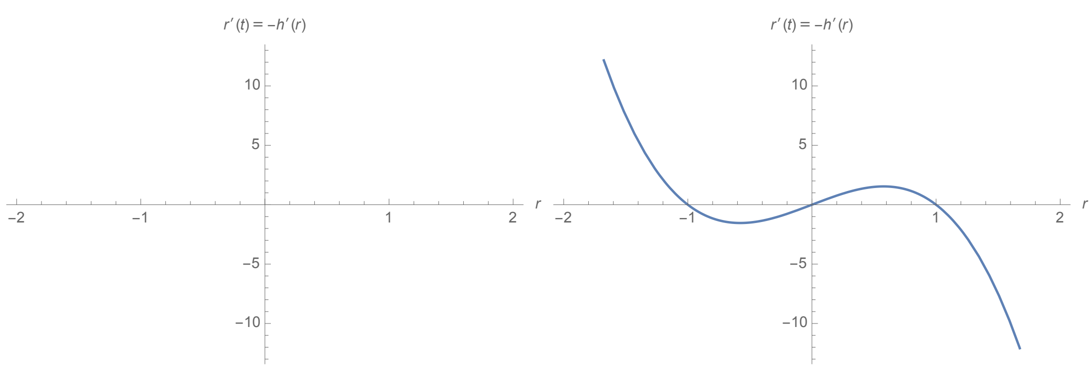
We draw the function, noting their relative values and shapes.



3.2 Slope of the hole!

Plot $\dot{r}$.

We examine the equation for $\dot{r}$ and plot it.



3.3 Steady r

Show on the preceding plots of $h(r)$ and $\dot{r}$ where $\dot{r}$=0. Are these stable or unstable to perturbations in r?

We indicate the zero crossings of $\dot{r}$ and the extrema of $h(r)$. The center point at $r = 0$ is unstable, the $|r| = 1$ locations are stable.

3.4 Round and Round

Now examine the $\dot{\theta}$ equation, which describes the motion of the pebble around the axis of the basin. What kind of motion does it describe?

Circles when on a steady r, or a point at $r = 0$. Otherwise, spirals.

3.5 Attractors

Describe the attractors of this system.

Unstable fixed point at $r = 0$, stable limit cycle at $|r| = 1$.

3.6 Angular Momentum?

Does this system conserve angular momentum $r^2\dot{\theta}$?

No, since there is no angular momentum near the origin, but the stone could emanate from there to arrive on the limit cycle.

3.7 Deformed and Deforming

If the motion of the pebble is partly to credit for the shape of the basin, how is this related to the dynamics of the pebble?

The stable attractor (limit cycle) of the pebble keeps it running around and around the deepest part of the basin, eroding it deeper and deeper.

4 Under Pressure

The momentum equation for a solid, liquid, or gas is.

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \sigma + \rho \mathbf{g}. \tag{25}$$

4.1 Stress Tensors

The difference between a solid, a liquid, and a gas stem from the stress tensor. Write the stress tensor σ_{ij} for each:

$$\sigma_{ij} = -p\delta_{ij} + S_{ijkl}\frac{1}{2}\left(\nabla_k \Delta x_l + \nabla_l \Delta x_k\right), \tag{26}$$

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right), \tag{27}$$

$$\sigma_{ij} = (-p + \lambda \nabla_k v_k)\delta_{ij} + \mu\left(\nabla_i v_j + \nabla_j v_i\right). \tag{28}$$

4.2 Pressure Only

For a liquid, solid or gas, if there is no velocity ($v = 0$) and no displacement ($\Delta x = 0$), there is only one balance possible in the momentum equation. What is it?

Plugging in, we find the hydrostatic balance,

$$0 = \nabla \cdot \sigma + \rho \mathbf{g} = -\nabla p + \rho \mathbf{g} \tag{29}$$

4.3 Solid Balance

If a solid is motionless ($v = 0$), but there has been a displacment ($\Delta x \neq 0$), is it possible for a solid to have a different steady balance than the previous one?

In this case, $0 = \nabla \cdot \sigma + \rho \mathbf{g}$, but there are stresses beyond the hydrostatic balance.

4.4 Fluid Balance

If a fluid is motionless ($v = 0$), but there has been a displacment ($\Delta x \neq 0$), is it possible for a fluid to have a different steady balance than the previous one?

No, since all of the deviatoric stress tensor depends on v, only the hydrostatic balance is motionless.

4.5 Once again into the Breach

Mathematically, why can you separate the part of the stress tensor that depends on the pressure from the rest of the stress tensor that depends on displacement or velocity?

Because the divergence is a linear operator, and the two parts of the stress tensor are just added together.